4. Chapter

Scale dependence of the True Score MTMM model

Germà Coenders, Joan M. Batista-Foguet and Albert Satorra

ABSTRACT This chapter starts by reviewing the consequences of scale dependence of Structural Equation Models. Next it shows the scale dependence of the True Score MTMM model presented in Chapter 1 and its practical consequences. Finally, it suggests alternative sets of non-linear constraints for the True Score MTMM model which make the model to be scale invariant without loss of parsimony. The scale dependent behaviour of the standard specification and the scale invariant behaviour of the alternative specifications are illustrated on a real data set.

Introduction

Scale invariance, as well as its absence, called scale dependence, has been described as a very relevant characteristic of Structural Equation Models (SEM). For a general overview of SEM we refer to Bollen (1989) or Jöreskog and Sörbom (1989). Swaminathan and Algina (1978) started the discussion of scale invariance and Cudeck (1989) made a review of the research on the topic. The consequences of scale dependence have been dealt with at length in the literature and can be particularly serious when SEM are fitted to correlation matrices instead of covariance matrices.

In this chapter, the topic will be introduced. The scale dependence of the True Score MTMM model presented in Chapter 1, which is a particular case of SEM, will then be discussed and alternative scale invariant specifications presented. The consequences of ignoring the scale dependence of the model in its standard specification and the scale invariant behaviour of the alternative specifications will be illustrated on a real data set.

In a Structural Equation Model, the covariance matrix $\Sigma$ of a set of observable variables is structured as a function $\Sigma=\Sigma(\pi)$ of a vector $\pi$ collecting the model parameters $\pi_s$ ($s=1,2,...,p$). Some of the elements of $\pi$ may be constrained. The covariance matrix $\Sigma$ is unaffected by the addition of arbitrary constants to the variables, but it does change when one or more variables are multiplied by arbitrary constants. When the $i$th variable is multiplied by a constant $\alpha_i$, its variance is multiplied by $\alpha_i^2$ and its covariance with any other variable is multiplied by $\alpha_i$. The definitions and terminology regarding scale invariance which follow are taken from Cudeck.

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A SEM $\Sigma = \Sigma(\pi)$ is said to be scale invariant if for any parameter vector $\pi$ and for any covariance matrix $\Sigma'$ which can be obtained from $\Sigma$ by multiplying one or more variables by arbitrary non-zero constants, there exists a parameter vector $\pi'$ which fulfils the constraints in $\pi$ such that $\Sigma' = \Sigma(\pi')$. Models which are not scale invariant are called scale dependent. For the models which are scale invariant, the concept of scale free parameter can be defined. A parameter $\pi$ is said to be scale free if $\pi_s = \pi'_s$.

These concepts are best explained by means of an example. Consider the Factor Analysis Model below, a simple particular case of SEM. The notation in Bollen (1989) and Jöreskog and Sörbom (1989) will be followed throughout this chapter.

$$y_1 = \lambda_{11} \eta_1 + \epsilon_1$$  
$$y_2 = \lambda_{21} \eta_1 + \epsilon_2$$  
$$y_3 = \lambda_{31} \eta_1 + \epsilon_3$$  

where $y_i$ is the $i$th observed variable; $\lambda_{ik}$ is the factor loading of the $i$th observed variable on the $k$th latent factor $\eta_k$, whose variance is $\psi_{kk}$; and $\epsilon_i$ is the measurement error term related to $y_i$, with variance $\theta_i$.

The specification of a factor analysis model is not complete without fixing the scale of the latent factor(s) by means of the so-called scaling constraints. This can be achieved by constraining one single loading to a non-zero value for each of the factors or, alternatively, by constraining the variance of the factor to a non-zero value. In our case we choose to constrain $\lambda_{11} = 1$. Given this, we can write the parameter vector as $\pi = (\lambda_{11} = 1, \lambda_{21}, \lambda_{31}, \psi_1, \theta_1, \theta_2, \theta_3)$.

The model assumes zero covariances among the error terms and between error terms and the factor. If these assumptions hold, the function $\Sigma = \Sigma(\pi)$, which relates the variances $\sigma_{ii}$ and covariances $\sigma_{ij}$ to the parameters of the model, can easily be derived by means of variance-covariance algebra as:

$$\sigma_{ij} = \psi_{ij} + \theta_{ij}$$  
$$\sigma_{22} = \lambda_{21}^2 \psi_{11} + \theta_{22}$$  
$$\sigma_{33} = \lambda_{31}^2 \psi_{11} + \theta_{33}$$  

If we multiply $y_1$, $y_2$ and $y_3$ by the arbitrary constants $\alpha_1$, $\alpha_2$ and $\alpha_3$ respectively, the expressions for the variances and covariances in $\Sigma'$ are:

$$\sigma'^{11} = (\psi_{11} + \theta_{11}) \alpha_1^2$$  
$$\sigma'^{22} = (\lambda_{21}^2 \psi_{11} + \theta_{22}) \alpha_2^2$$  
$$\sigma'^{33} = (\lambda_{31}^2 \psi_{11} + \theta_{33}) \alpha_3^2$$  

$$\sigma'^{21} = \lambda_{21} \psi_{11} \alpha_1 \alpha_2$$  

(3)
Scale dependence of the True Score MTMM model

\[ \sigma^*_{31} = \lambda_{31} \psi_{11} \alpha_1 \alpha_3 \]
\[ \sigma^*_{32} = \lambda_{32} (\lambda_{31} \psi_{11} \alpha_2 \alpha_3) \]

thus, the expression \( \Sigma^* = \Sigma(\pi^*) \) can be written as:

\[
\begin{align*}
(\psi_{11} + \theta_{11}) \alpha_2 &= \psi_{11} + \theta_{11} \\
(\lambda_{21} \psi_{11} + \theta_{22}) \alpha_3 &= \lambda_{21} \psi_{11} + \theta_{22} \\
(\lambda_{31} \psi_{11} + \theta_{33}) \alpha_3 &= \lambda_{31} \psi_{11} + \theta_{33}
\end{align*}
\]

where the terms to the right of the equal sign express the structure (2) of the original model in terms of the parameter vector \( \pi^* = (\lambda_{11}^*, 1, \lambda_{21}, \lambda_{31}, \psi_{11}^*, \theta_{11}, \theta_{22}, \theta_{33}) \).

The model will be scale invariant if the system of equations (4) has a solution for the elements in \( \pi^* \). For this model, it can be shown that the solution exists and is:

\[
\pi^* = (\lambda_{11}^* = 1, \lambda_{21}, \lambda_{31}, \psi_{11}^*, \theta_{11}, \theta_{22}, \theta_{33})
\]

Thus, the model in (1) is scale invariant. Note that this particular model has no scale free parameters as, except \( \lambda_{11} \), all elements in \( \pi^* \) differ from those in \( \pi \).

Swaminathan and Algina (1978) and Cudeck (1989) noted that Factor Analysis models in which more constraints are imposed (on the factor loadings or the factor variances) than those strictly needed for scaling are usually scale dependent, except if the constraints consist in fixing factor loadings to the value of zero.

In order to illustrate this point let us consider a model which is similar to the model (1) but introduces the additional constraints \( \lambda_{21} = 1 \) and \( \lambda_{31} = 1 \). In order that the alternative set of parameters in (5) continues to accommodate the variances and covariances in (3) while fulfilling the additional constraints specified in the model, it is necessary that both \( \lambda_{32} \alpha / \alpha_1 \) and \( \lambda_{31} \alpha / \alpha_1 \) be equal to 1, i.e. it is necessary that \( \alpha_2 = \alpha_3 = \alpha_1 \). Thus, \( \pi^* \) only exists for a limited set of \( \alpha \) constants, whereby it is proven that the model with all loadings equal to 1 is scale dependent. Scale dependence implies that multiplying any variable(s) by arbitrary constants may actually change the model which the researcher is trying to fit. In other words, the constraints in the model may change its meaning and implications, so that the model may hold when the variables are measured with certain units and fail to hold when the units of measurement of some variables change. The \( \chi^2 \) goodness of fit test statistic (see for instance Bollen, 1989) may also change.

In applied research, SEM are often fitted to correlation matrices instead of covariance matrices. The step from covariances to correlations implies multiplying each variable by the
inverse of its standard deviation and thus, it can change the meaning of a scale dependent model. In fact, even after standardisation, the parameter estimates obtained from the covariance matrix will not be equal to those obtained from the correlation matrix.

The fit of scale dependent SEM's to correlation matrices can result in misleading interpretations of the model's constraints and in the undesirable consequence that two researchers fitting the same model on the same data but analysing different matrices can reach contradictory conclusions when interpreting the $\chi^2$ goodness of fit test statistic.

For all the above reasons, Cudeck (1989) recommends either analysing only covariance matrices or using only scale invariant models. In this chapter we shall examine both suggestions regarding the True Score MTMM model specified in Chapter 1. First we will illustrate the scale dependence of the model and examine the requirements for the covariance matrix to be analysed, in particular, which changes of scale (if any) are allowed. Next, we will suggest alternative specifications for the model, for which scale invariance holds and the scale of the variables no longer matters.

As regards the individual model parameters, Cudeck points out that when a correlation matrix is analysed, most standard software for SEM can correctly estimate the standard errors only for the scale free parameters. This occurs because the standardisation of the variables involves using a set of random $\alpha$ values (the inverse of the standard deviations of the variables). In order to get correct standard errors of all parameters, specific software for correlation structure analysis should be used. For this reason, Cudeck recommends that scale invariant models should be specified to contain as many scale free parameters as possible.

Scale dependence of the True Score MTMM Model. Suggestions regarding the matrix to be analysed

This model was suggested in Saris and Andrews (1991) as a modification of earlier models (Althauser et al., 1971; Alwin, 1974; Werts & Linn, 1970; Jöreskog, 1971) and used in Scherpenzeel (1995) with the set of constraints which were considered in Chapter 1. The model assumes a facet measurement design (Campbell & Fiske, 1959) in which a set of $t$ factors (traits) are all measured with the same set of $m$ measurement procedures (methods). For a usual design with $t=3$ traits and $m=3$ methods, the model is specified as a Second Order Factor Analysis Model with the following 18 equations:

\begin{align*}
y_1 &= \lambda_{11} \eta_1 + \epsilon_1; & \eta_1 &= \gamma_{11} \xi_{11} + \gamma_{12} \xi_{12} \\
y_2 &= \lambda_{22} \eta_2 + \epsilon_2; & \eta_2 &= \gamma_{21} \xi_{21} + \gamma_{22} \xi_{22} \\
y_3 &= \lambda_{33} \eta_3 + \epsilon_3; & \eta_3 &= \gamma_{31} \xi_{31} + \gamma_{32} \xi_{32} \\
y_4 &= \lambda_{44} \eta_4 + \epsilon_4; & \eta_4 &= \gamma_{41} \xi_{41} + \gamma_{42} \xi_{42} + \gamma_{43} \xi_{43} \\
y_5 &= \lambda_{55} \eta_5 + \epsilon_5; & \eta_5 &= \gamma_{51} \xi_{51} + \gamma_{52} \xi_{52} + \gamma_{53} \xi_{53} \\
y_6 &= \lambda_{66} \eta_6 + \epsilon_6; & \eta_6 &= \gamma_{61} \xi_{61} + \gamma_{62} \xi_{62} + \gamma_{63} \xi_{63}
\end{align*}
Scale dependence of the True Score MTMM model

\[
\begin{align*}
    y_7 &= \lambda_{77} \eta_7 + \varepsilon_7; & \eta_7 &= \gamma_{71} \xi_{t1} + \gamma_{76} \xi_{m6} \\
    y_8 &= \lambda_{88} \eta_8 + \varepsilon_8; & \eta_8 &= \gamma_{82} \xi_{t2} + \gamma_{86} \xi_{m6} \\
    y_9 &= \lambda_{99} \eta_9 + \varepsilon_9; & \eta_9 &= \gamma_{93} \xi_{t3} + \gamma_{96} \xi_{m6}
\end{align*}
\]

Two types of equations are involved in (6). In the left hand set of equations, each \( y_i \) is related to its corresponding first order true score factor \( \eta_i \), which is interpreted as the stable part of \( y_i \). \( \lambda_{ii} \) is the factor loading of \( y_i \) and \( \varepsilon_i \) is its measurement error term, with variance \( \theta_{ii} \).

The right set of equations in (6) assumes that \( y_1 \) to \( y_3 \) are measured with method 1, \( y_4 \) to \( y_6 \) with method 2, and \( y_7 \) to \( y_9 \) with method 3. This set of equations relates each true score factor to one second order trait factor (through the \( \gamma \) parameters, called trait effects) and to one second order method factor (through the \( \gamma \) parameters, called method effects). \( \xi_{t1} \) to \( \xi_{t3} \) are the trait factors, with covariances \( \phi_{t21}, \phi_{t31}, \phi_{t32} \); and \( \xi_{m4} \) to \( \xi_{m6} \) are the method factors.

The model assumes zero covariances among method factors, between trait factors and method factors, among measurement error terms, and between error terms and factors of any type. Under these assumptions, the percentage of variance of \( y \) explained by \( \eta \) (squared standardised \( \lambda \)) can be interpreted as reliability. The percentage of variance of \( \eta \) explained by the trait factor (squared standardised \( \gamma \)) can be interpreted as validity. The percentage of variance of \( \eta \) explained by the method factor (squared standardised \( \gamma \)) can be interpreted as invalidity. See in this respect Saris and Andrews (1991).

In Scherpenzeel (1995), as well as in Chapter 1, the scale of the first order true score factors is specified by letting all \( \lambda_{ii} \) loadings be equal to 1 and the scale of the second order trait factors is specified by letting their variance be equal to 1. The variances of the method factors constitute unconstrained model parameters (named \( \phi_{m44}, \phi_{m55}, \phi_{m66} \)) but in addition the model specifies the constraints:

\[
\gamma_{14} = \gamma_{24} = \gamma_{34} = \gamma_{45} = \gamma_{56} = \gamma_{67} = 1
\]

(7)

whereas just constraining one of the \( \gamma \)'s for each method would have been enough for scaling purposes. With this specification, the model has a total of 24 parameters and 21 degrees of freedom when \( t=3 \) and \( m=3 \).

It can be shown that the constraints in (7) lead to a scale dependent True Score MTMM model. When multiplying the \( y_i \) variables by a set of \( \alpha \) constants, the model, the interpretation of its constraints and its goodness of fit will in principle change. This will constitute a problem if the scale of any variable is changed, or, what is more common, if the variables are standardised, i.e. if the analysis is based upon the correlation matrix.
An interesting particular case is when:

\[ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 \quad (8) \]

\[ \alpha_7 = \alpha_8 = \alpha_9 \]

i.e. when all variables measured with the same method are rescaled in the same way. It can be shown that when this is the case, the constraint of equality of method effects in (7) has the same implications on the covariance structure of the original variables as on that of the rescaled ones, so that neither the model nor its goodness of fit change.

This feature is quite interesting. As has been illustrated throughout this book, methods very often consist of different response scales in questionnaires, which may have different ranges. If a researcher wants all methods to yield comparable units of measurement, he or she can linearly transform the responses so that the allowed response range is the same for all methods. This will actually involve rescaling the variables with a set of constants which fulfill (8).

Proving the scale dependence of the model and deriving the condition (8) are rather tedious exercises. Instead of developing them here we will provide an illustration based on real data. For the illustrations in this chapter we use a real \( t=3, m=3 \) MTMM data set which contains evaluations of self-perceived life satisfaction obtained in a survey of the Catalan population carried out in 1989. As traits, we consider three domains of life satisfaction:

- **hou**: housing (\( \xi_{1t} \)).
- **fin**: financial situation (\( \xi_{2t} \)).
- **con**: social contacts (\( \xi_{3t} \)).

All three traits are measured with three response scales ranging from completely dissatisfied to completely satisfied:

- **100**: numeric scale ranging from 0 to 100 (\( \xi_{1m} \)).
- **5**: 5-point scale with all-labelled categories ranging from 1 to 5 (\( \xi_{2m} \)).
- **11**: 11-point scale ranging from 0 to 10 (\( \xi_{3m} \)).

Details on the questionnaire and data collection can be found in Batista-Foguet et al. (1996). Sample size is 406. The covariance matrix is given in Table 1.

We estimated the True Score MTMM model presented in this section [i.e. including the constraints in (7)] by Maximum Likelihood (ML) using the program LISREL8 (Jöreskog & Sörbom 1989, 1993). The input file for the program LISREL8 which was used to analyse the matrix in Table 1 can be seen in the appendix. The estimation was repeated 3 times using the following input matrices:
The covariance matrix among the original variables as shown in Table 1.

A covariance matrix computed from rescaled variables in order that all methods range from 0 to 100. The change of scale was made as follows: in responses using the second method (with 5 points and range=4), 1 was subtracted and the result was multiplied by 25; responses using the third method (with 11 points and range=10) were multiplied by 10.

The Pearson correlation matrix among the original variables.

**Table 1.**

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<th>fin100</th>
<th>con100</th>
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Table 2 shows the unstandardised and standardised loading, method effect and trait effect estimates, the trait correlation estimates and the $\chi^2$ goodness of fit test statistic obtained from each of the three input matrices. Error and method variances are omitted for the sake of simplicity.

As expected, the rescaling of all responses to lie between 0 and 100 (second set of estimates in Table 2) has neither affected the model fit nor the way the model behaves. All standardised estimates are exactly the same as when the original data are used (first set of estimates). The unstandardised trait effects can be derived from the ones obtained from the original data simply by multiplying them by 25 (for variables using the 5-point scale) or by 10 (for variables using the 11-point scale).

This rescaling makes it possible to better interpret and compare the unstandardised estimates. Observe for instance, that when the original data are used, the unstandardised trait effects vary a great deal across methods due to the different units of measurement. This does not occur when the rescaled data are used.

The analysis of the correlation matrix (third set of estimates in Table 2), reveals the scale dependence of the model. Note that the behaviour of the model has changed with respect to the two previous sets of estimates. Not only have the unstandardised estimates changed, as happens with any model when the scale is changed, but the standardised estimates and the
goodness of fit statistic have also changed. In this particular case, the changes are small and do not lead to different overall conclusions, but this does not need to be the case in all data sets.

As appealing as it may be for substantive reasons (Saris & Andrews, 1991; Scherpenzeel, 1995), this model should better not be fitted to correlation matrices, at least with the set of constraints considered in this section. However, the model may be perfectly appropriate for the analysis of covariance matrices, even if certain meaningful changes of scale are performed on the variables, provided that the changes of scale fulfill the condition (8).

Table 2. Estimates of the scale dependent True Score MTMM model

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<th>Standardised</th>
<th>Trait Correlations</th>
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Alternative scale invariant specifications of the model which allow for the analysis of both covariance and correlation matrices under arbitrary scales of the variables are indeed possible and they are dealt with in the next section.

Scale dependence of the True Score MTMM Model. Suggestions regarding alternative scale invariant specifications

In this section we present a scale invariant True Score MTMM model. It can be shown that the constraints in (7) are responsible for the scale dependence of the model. A first straightforward way to produce a scale invariant specification is simply to remove these constraints. Unfortunately, this may have several drawbacks.

A first rather technical drawback is that this alternative parametrization is less parsimonious. In particular, when \( i=3 \) and \( m=3 \) it implies an additional 6 parameters to be estimated. Heavily parametrized models are reported to be problematic in the literature, as they can lead to non-convergent or non-sense estimates (Rindskopf, 1984), particularly if they contain a large number of factors relative to the number of observed variables, as the True Score MTMM model does. We did fit this less restrictive model on the covariance matrix in Table 1, and we obtained an \( \chi^2 \) fit statistic of 44.45 which, in comparison with the figure of 45.76 reported in Table 2 and with the loss of 6 degrees of freedom, suggests that it is hardly worthwhile to release the constraint (7) in this particular case.

A second drawback is a rather substantive one. The literature links method variance and method effects to the method of measurement, so that the equal impact of the method for all measurements sharing it is often a very meaningful assumption, which is worth building into the models (Andrews, 1984; Saris, 1990).

### Table 2 (continued)

Estimates based on the correlation matrix among original variables

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<tr>
<td>hou5</td>
<td>1 0.836</td>
<td>1 0.856</td>
<td>0.965</td>
</tr>
<tr>
<td>fin5</td>
<td>1 0.898</td>
<td>1 0.913</td>
<td>0.969</td>
</tr>
<tr>
<td>con5</td>
<td>1 0.822</td>
<td>1 0.854</td>
<td>0.963</td>
</tr>
<tr>
<td>hou11</td>
<td>1 0.900</td>
<td>1 0.962</td>
<td>0.941</td>
</tr>
<tr>
<td>fin11</td>
<td>1 0.892</td>
<td>1 0.954</td>
<td>0.940</td>
</tr>
<tr>
<td>con11</td>
<td>1 0.822</td>
<td>1 0.897</td>
<td>0.931</td>
</tr>
</tbody>
</table>
Therefore, in the remainder of this chapter we will restrict ourselves to alternative specifications which involve no loss of parsimony and which somehow impose constraints which can be interpreted as an equal impact of the method on all measurements using it.

The solution we suggest is quite simple in concept. Instead of constraining method effects to be equal within a method, we will constrain invalidities to be equal within a method. In the True Score MTMM model validity and invalidity must add up to one. Therefore, validities will also be constrained to be equal within a method. The rationale for this choice is quite intuitive. Method and trait effects are dependent on the scale of the variables, and therefore their values and the interpretation of their constraints will change if the scales are changed. Validities and invalidities, on the contrary, are percentages of explained variance and can be interpreted in the same way regardless of the scale. Two alternative sets of constraints can be used with this aim.

A first set of constraints (which leads to what we call parametrization with proportional method effects) restricts the method effects $\gamma_m$ to be proportional to the trait effects $\gamma_t$. One $\gamma_m$ is fixed to 1 for each method and the remaining ones depend on the $\gamma_t$ value. The non-linear constraints below are used instead of the linear ones in (7):

$$
\begin{align*}
\gamma_{m14} &= \gamma_{m55} = \gamma_{m76} = 1 \\
\gamma_{m24} &= \gamma_{t22} / \gamma_{t11} \\
\gamma_{m34} &= \gamma_{t33} / \gamma_{t11} \\
\gamma_{m55} &= \gamma_{t52} / \gamma_{t41} \\
\gamma_{m65} &= \gamma_{t63} / \gamma_{t41} \\
\gamma_{m86} &= \gamma_{t82} / \gamma_{t71} \\
\gamma_{m96} &= \gamma_{t93} / \gamma_{t71} 
\end{align*}
$$

The intuitive idea underlying this first parametrization is the following: if a variable is multiplied by a constant, its trait effect should be multiplied by the same constant. If the method effect is not free to change, as in the standard specification, then scale dependence arises. If the method effect is forced to change in the same proportion as the trait effect, not only do scale dependence problems disappear, but also the percentages of trait and method variance (validity and invalidity) remain unchanged after the change of scale and are constant within a method.

A second set of constraints (which leads to what we call parametrization with unit true score variance) makes the variance of the true scores to be equal to 1 while keeping the restriction of all-unit method effects in (7). This implies that the squared trait effect $\gamma_t^2$, plus the method variance $\phi_m$ must equal 1, which can be formulated through the following non-linear constraints:
Scale dependence of the True Score MTMM model

\[ \phi_{m44} = 1 - \gamma_{t11} \]
\[ \phi_{m55} = 1 - \gamma_{t41} \]
\[ \phi_{m66} = 1 - \gamma_{t71} \]

(10)

with the specification of the \( \lambda \)'s, as free model parameters which is necessary due to the unit variance of the true score factors.

The intuitive ideas underlying this second parametrization are the following: First, the units of measurement do not affect the trait and method effects, but the loadings of the variables on the true scores (\( \lambda \)'s), which, being unconstrained, can accommodate any change of scale. Second, since both trait factors and true score factors have unit variance, the squared trait effect can be directly interpreted as validity. Thus, the equality of trait effects within a method implied in (10) actually means equality of validities within a method. This parametrization is similar to one suggested in Dudgeon (1994).

In spite of the quite different sets of constraints involved, both the parametrization with proportional trait loadings and with the one with unit true score variances are equivalent. For any set of parameters of either parametrization we can find a set of parameters for the other one, which yields the same implied variances-covariances (see for instance Luijben, 1989 for formal definitions of equivalence). Moreover, when standardised, the parameters take the same value for both parametrizations. It should also be noted that both parametrizations are as parsimonious as the standard one but not equivalent to it. Both parametrizations have 21 degrees of freedom when \( t=3 \) and \( m=3 \).

Both parametrizations can be formulated in the program LISREL8 which, as a new feature with respect to earlier versions, allows the user to introduce certain forms of non-linear constraints. We will illustrate the behaviour of both parametrizations by fitting them to the data in Table 1 by ML using the LISREL8 input files presented in the appendix. The estimates using the parametrization with proportional method effects are in Table 3, and use both covariances and correlations as input. The estimates using the parametrization with unit true score variances are in Table 4.
Table 3.
Estimates of the scale invariant True Score MTMM model. Parametrization with proportional method effects

<table>
<thead>
<tr>
<th></th>
<th>Unstandardised</th>
<th>Standardised</th>
<th>Trait Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_{ii}$</td>
<td>$\gamma_{it}$</td>
<td>$\gamma_{itk}$</td>
</tr>
<tr>
<td>hou100</td>
<td>1 20.057</td>
<td>1.000</td>
<td>0.839</td>
</tr>
<tr>
<td>fin100</td>
<td>1 20.925</td>
<td>1.043</td>
<td>0.908</td>
</tr>
<tr>
<td>con100</td>
<td>1 18.624</td>
<td>0.929</td>
<td>0.840</td>
</tr>
<tr>
<td>hou5</td>
<td>1 0.872</td>
<td>1.000</td>
<td>0.854</td>
</tr>
<tr>
<td>fin5</td>
<td>1 0.939</td>
<td>1.076</td>
<td>0.914</td>
</tr>
<tr>
<td>con5</td>
<td>1 0.778</td>
<td>0.892</td>
<td>0.854</td>
</tr>
<tr>
<td>hou11</td>
<td>1 2.088</td>
<td>1.000</td>
<td>0.963</td>
</tr>
<tr>
<td>con11</td>
<td>1 1.763</td>
<td>0.844</td>
<td>0.898</td>
</tr>
<tr>
<td>fin11</td>
<td>1 2.047</td>
<td>0.980</td>
<td>0.953</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Unstandardised</th>
<th>Standardised</th>
<th>Trait Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_{ii}$</td>
<td>$\gamma_{it}$</td>
<td>$\gamma_{itk}$</td>
</tr>
<tr>
<td>hou100</td>
<td>1 0.781</td>
<td>1.000</td>
<td>0.839</td>
</tr>
<tr>
<td>fin100</td>
<td>1 0.840</td>
<td>1.076</td>
<td>0.908</td>
</tr>
<tr>
<td>con100</td>
<td>1 0.810</td>
<td>1.037</td>
<td>0.840</td>
</tr>
<tr>
<td>hou5</td>
<td>1 0.834</td>
<td>1.000</td>
<td>0.854</td>
</tr>
<tr>
<td>fin5</td>
<td>1 0.891</td>
<td>1.068</td>
<td>0.914</td>
</tr>
<tr>
<td>con5</td>
<td>1 0.830</td>
<td>0.995</td>
<td>0.854</td>
</tr>
<tr>
<td>hou11</td>
<td>1 0.896</td>
<td>1.000</td>
<td>0.963</td>
</tr>
<tr>
<td>con11</td>
<td>1 0.839</td>
<td>0.936</td>
<td>0.953</td>
</tr>
</tbody>
</table>

A first look at both tables reveals that the standardised estimates, the trait correlations, and the $\chi^2$ statistic are equal regardless of the parametrization and the type of matrix analysed, thus showing that both parametrizations are equivalent, that both are scale invariant and that both include scale free trait factor correlations. Both parametrizations can then be used for the analysis of both correlation and covariance matrices. Note also that the standardised $\gamma$'s and $\gamma$'s are equal within a method for both parametrizations. The unstandardised estimates, do of course, change from parametrization to parametrization due to the different scale of the true score factors; and they also change from matrix to matrix, as can be expected from any change of scale of the observed variables.
Table 4.
Estimates of the scale invariant True Score MTMM model. Parametrization with unit true score variance

<table>
<thead>
<tr>
<th></th>
<th>Unstandardised</th>
<th>Standardised</th>
<th>Trait Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_{ii}$</td>
<td>$\gamma_{ik}$</td>
<td>$\gamma_{mk}$</td>
</tr>
<tr>
<td>hou100</td>
<td>21.173</td>
<td>0.947</td>
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<tr>
<td>fin100</td>
<td>22.089</td>
<td>0.947</td>
<td>1.000</td>
</tr>
<tr>
<td>con100</td>
<td>19.660</td>
<td>0.947</td>
<td>1.000</td>
</tr>
<tr>
<td>hou5</td>
<td>0.903</td>
<td>0.966</td>
<td>1.000</td>
</tr>
<tr>
<td>fin5</td>
<td>0.972</td>
<td>0.966</td>
<td>1.000</td>
</tr>
<tr>
<td>con5</td>
<td>0.805</td>
<td>0.966</td>
<td>1.000</td>
</tr>
<tr>
<td>hou11</td>
<td>2.226</td>
<td>0.938</td>
<td>1.000</td>
</tr>
<tr>
<td>fin11</td>
<td>2.182</td>
<td>0.938</td>
<td>1.000</td>
</tr>
<tr>
<td>con11</td>
<td>1.879</td>
<td>0.938</td>
<td>1.000</td>
</tr>
</tbody>
</table>

$\chi^2$ fit statistic
hou11 0.955 0.938 1.000 0.963 0.938 0.346
fin11 0.939 0.938 1.000 0.953 0.938 0.346

In spite of the equivalence of both parametrizations, the one with unit true score variance has an additional advantage: the $\gamma_{ik}$’s are scale free parameters. This means that even when unstandardised they can be interpreted as the square root of validity and that even standard software can produce their correct standard errors.

When the parametrization with unit true score variance is fitted to the correlation matrix, it could be expected that the unstandardised $\lambda_{ii}$’s equal the standardised ones but Table 4 shows this not to be the case. This occurs because this type of models can have diagonal residuals, so that when analysing the correlation matrix, the variances implied by the model parameters do not equal 1. Cudeck (1989) argues that there is nothing wrong with this kind of models and he only advises that unstandardised estimates should be rescaled in order to yield unit implied variances before they are interpreted. This is precisely what the standardisation option in LISREL8 (SC or Standardise Completely) does, so that we advise always using this option.
when analysing a correlation matrix.

Discussion

This chapter has shown that the True Score MTMM model specified in Chapter 1 is scale dependent. This constitutes no drawback when a covariance matrix is analysed, even if the scales are changed in a consistent way within the methods. However problems do arise when the model is fitted to a correlation matrix or when the results of several researchers using different types of matrix are to be compared. This leads to two alternative feasible strategies. The first one is to always analyse a covariance matrix and the second to seek an alternative, though meaningful, scale invariant model.

This chapter presents two alternative parametrizations of a scale invariant model which imply no loss of parsimony and which are appealing from a substantive point of view as they imply that validity depends on the method and reliability on the combination of trait and method, in coherence with Andrews' interpretations. They are also appealing because it can be shown that both parametrizations lead to additive method effects as defined in Coenders and Saris (1996), and are equivalent to the scale invariant additive Correlated Uniqueness model specified by these authors. The parametrization with unit variance true score factors is particularly appealing because it makes validity-related parameters scale-free. A drawback to the alternative parametrizations is that they involve non-linear constraints, but nowadays even widely available standard software allows for them.

References


Appendix. LISREL8 input files

True Score MTMM model in Chapter 1 including the constraint in (7):

```
TI SCALE DEPENDENT TRUE SCORE MTMM MODEL
DA NI=9 NO=406 MA=CM
LA
HOU100 FIN100 CON100 HOUS5 FINS5 CONS5 HOU11 FIN11 CON11
CM SY FI=SAT.COV
MO NY=9 NE=9 NK=6 LI=FU,FI GA=FU,FI PH=SY,FI PS=ZE TE=FU,FI
LK
HOU FIN CON '100P' '5P' '11P'
LE
THOU100 TFN100 TCON100 THOU5 TFINS5 TCONS5 THOU11 TFN11 TCON11
VA 1 LY(1,1) LY(2,2) LY(3,3) LY(4,4) LY(5,5) LY(6,6) LY(7,7) LY(8,8) LY(9,9)
FR TE(1,1) TE(2,2) TE(3,3) TE(4,4) TE(5,5) TE(6,6) TE(7,7) TE(8,8) TE(9,9)
FR GA(1,1) GA(2,2) GA(3,3) GA(4,1) GA(5,2) GA(6,3) GA(7,1) GA(8,2) GA(9,3)
FR PH(2,1) PH(3,1) PH(3,2) PH(4,4) PH(5,5) PH(6,6)
VA 1 PH(1,1) PH(2,2) PH(3,3)
ST 00 GA(1,1) GA(2,2) GA(3,3)
ST 05 GA(4,1) GA(5,2) GA(6,3)
ST 10 PH(1,1) PH(2,2) PH(3,3)
OU NS ML NO=3 SC
```

Scale invariant parametrization of the True Score MTMM model with proportional method effects:

```
TI SCALE INVARIANT TRUE SCORE MTMM MODEL WITH PROPORTIONAL METHOD EFFECTS
DA NI=9 NO=406 MA=CM
LA
HOU100 FIN100 CON100 HOUS5 FINS5 CONS5 HOU11 FIN11 CON11
CM SY FI=SAT.COV
MO NY=9 NE=9 NK=6 LI=FU,FI GA=FU,FI PH=SY,FI PS=ZE TE=FU,FI
LK
HOU FIN CON '100P' '5P' '11P'
LE
THOU100 TFN100 TCON100 THOU5 TFINS5 TCONS5 THOU11 TFN11 TCON11
VA 1 LY(1,1) LY(2,2) LY(3,3) LY(4,4) LY(5,5) LY(6,6) LY(7,7) LY(8,8) LY(9,9)
FR TE(1,1) TE(2,2) TE(3,3) TE(4,4) TE(5,5) TE(6,6) TE(7,7) TE(8,8) TE(9,9)
FR GA(1,1) GA(2,2) GA(3,3) GA(4,1) GA(5,2) GA(6,3) GA(7,1) GA(8,2) GA(9,3)
VA 1 GA(1,1) GA(2,2) GA(3,3)
FR GA(4,4) GA(5,5) GA(6,6)
VA 1 PH(1,1) PH(2,2) PH(3,3)
ST 00 GA(1,1) GA(2,2) GA(3,3)
ST 05 GA(4,1) GA(5,2) GA(6,3)
ST 10 PH(1,1) PH(2,2) PH(3,3)
OU NS ML NO=3 SC
```

\[
\text{CO GA}(2,4) = \text{GA}(2,2) \cdot \text{GA}(1,1)^{-1}.
\]

\[
\text{CO GA}(3,4) = \text{GA}(3,2) \cdot \text{GA}(1,1)^{-1}.
\]

\[
\text{CO GA}(5,5) = \text{GA}(5,2) \cdot \text{GA}(1,1)^{-1}.
\]

\[
\text{CO GA}(6,6) = \text{GA}(8,2) \cdot \text{GA}(7,1)^{-1}.
\]

\[
\text{CO GA}(9,6) = \text{GA}(9,3) \cdot \text{GA}(7,1)^{-1}.
\]

OU NS ML NO=3 SC
Scale invariant parametrization of the True Score MTMM model with unit true score variance:

```
TI SCALE INVARIANT TRUE SCORE MTMM MODEL WITH UNIT TRUE SCORE VARIANCE
DA NI=9 NO=406 MA=CM
LA
HOU100 FIN100 CON100 HOU5 FIN5 CON5 HOU11 FIN11 CON11
CM SY FI=SAT.COV
MO NY=9 NK=6 LY=FU,FI GA=FU,FI PH=SY,FI PS=ZE TE=FU,FI
LR
HOU FIN CON '100P' '5P' '11P'
LE
THOU100 TFIN100 TCON100 THOU5 TFIN5 TCON5 THOU11 TFIN11 TCON11
FR
LY(1,1) LY(2,2) LY(3,3) LY(4,4) LY(5,5) LY(6,6) LY(7,7) LY(8,8) LY(9,9)
FR
TE(1,1) TE(2,2) TE(3,3) TE(4,4) TE(5,5) TE(6,6) TE(7,7) TE(8,8) TE(9,9)
FR
GA(1,1) GA(2,2) GA(3,3) GA(4,1) GA(5,2) GA(6,3) GA(7,1) GA(8,2) GA(9,3)
FR
PH(2,1) PH(3,1) PH(3,2) PH(4,4) PH(5,5) PH(5,6)
VA 1 PH(1,1) PH(2,2) PH(3,3)
ST 20 LY(1,1) LY(2,2) LY(3,3)
ST 17 LY(4,4) LY(5,5) LY(6,6) TE(7,7) TE(8,8) TE(9,9)
ST 20 LY(7,7) LY(8,8) LY(9,9)
ST 19 GA(1,1) GA(2,2) GA(3,3) GA(4,1) GA(5,2) GA(6,3) GA(7,1) GA(8,2) GA(9,3)
ST 3 PH(2,1) PH(3,1) PH(3,2)
ST 2 PH(4,4) PH(5,5) PH(5,6) TE(4,4) TE(5,5) TE(6,6)
ST 200 TE(1,1) TE(2,2) TE(3,3)
CO PH(4,4)=1-GA(4,1)**2
CO PH(5,5)=1-GA(5,2)**2
EQ GA(1,1) GA(2,2) GA(3,3)
EQ GA(4,1) GA(5,2) GA(6,3)
EQ GA(7,1) GA(8,2) GA(9,3)
OU NS ML ND=3 SC
```