Mathematical Optimization for Economics

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In this short note, we recall by way of graphical representations three useful tools that the student of economics should always keep in mind: optimization under restrictions relative to the choice variable, regime change and comparative statics. They are exposed on Figures 1, 2 and 3 respectively.

1 Optimization Under Restrictions

In a typical economic problem, an agent cares for some dimension \( y \) such as profit or utility that is determined by some variable \( x \) such as quantity, price or quality through the relationship \( y = f(x) \). Figure 1 displays this relationship for a number of configurations. We will learn to distinguish a mathematical optimum from an economic one.

The unconstrained maximum of \( f \) is achieved at \( x_0 \), the solution to the first order condition \( f'(x) = 0 \) (FOC) i.e., when the curve becomes flat. This mathematical optimum is NOT the economic optimum, it is only a candidate because in most situations, the decision maker faces restrictions such as \( x \geq x_1 \) and \( x \leq x_2 \) that arise from his interaction with other economic agents or with the market.

On panel 1, these conditions define two stripped areas of allowed values; their intersection, the shaded area, is the economic domain from which the decision maker can pick a decision. Since \( x_0 \) belongs to this domain, the economic optimum is \( x^* = x_0 \). On panels 2 and 3, the economic domain does not contain the mathematical optimum so that the economic optimum is a corner solution, respectively.

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the maximum and minimum allowed levels for the endogenous variable ($x^* = x_2$ resp. $x_1$). Lastly, panel 4 displays the case where the restrictions are incompatible among themselves and leave no way for the agent to make a pick. His optimization problem, in that case, has no solution. In retrospect, we may say that the economic problem was badly formulated.

## 2 Regime Change

In economic analysis, a decision maker may face completely different environments according to whether she follows a high or low strategy; we say that she stumbles upon a change of regime. This means that the ultimate outcome is determined by a different process according to whether the endogenous variable is high or low. Figure 2 illustrates such an optimization problem.

Concretely, the objective assumes two distinct forms according to the domain where the endogenous variable is picked. We have $f(x) = g(x)$ if $x \leq x_0$ while $f(x) = h(x)$ if $x \geq x_0$, where $x_0$ is the threshold for regime switching. The economic curve
$f(.)$ is shown as a plain curve. We also draw the entire mathematical curves $g(.)$ and $h(.)$ as well as their maximum arguments $x_g$ and $x_h$.

![Figure 2: Mathematical Optimization](image)

On panel 5, we see that $x^*$ the (economic) maximum of $f$ coincides with $x_g$ the (mathematical) maximum of $g$. The fact that $x_h < x_0$ simply means that on the economic domain where $f = h$, which is $x \geq x_0$, the objective decreases with the variable so that we tend to leave that domain in order to find the maximum of $f$. On panel 6, the reverse happens: $f$ is increasing over the entire domain $x \leq x_0$, so that we leave it and find the (economic) maximum of $f$ where $h$ is maximum. Panel 7 is a corner solution because the shapes of $g$ and $h$ are such that we do not want to pick $x < x_0$ nor $x > x_0$, thus the frontier is the optimum. Lastly, panel 8 displays the case where both $g$ and $f$ achieve interior maxima. This means that $f$ has to local maximizers and the only way to pick among them is to compare $f(\bar{x})$ with $f(x^*)$. 

3
3 Comparative Statics

Given two ranked functions $g < h$, a solution $x_g$ to $g = z$ ($z$ being a parameter) is always lesser than a solution $x_h$ to $h = z$ for otherwise the curves would have to cross. When we take weighted averages of these basics, the solution lies in between and given that $\alpha > \beta$, we have $x_f < x_k$. The typical application will involve marginal and average cost function which are ranked when the technology displays either economies or diseconomies of scale.

\begin{equation}
\begin{align*}
f &= \alpha g + (1 - \alpha) h \\
k &= \beta g + (1 - \beta) h
\end{align*}
\end{equation}

Figure 3: Mathematical Optimization